

Classification of directed regular Ricci-flat graphs

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Abstract

In this paper, we define the Ricci curvature on directed graphs modifying Lin-Lu-Yau's Ricci curvature on undirected graphs. As a main result, we classify directed regular graphs with flat Ricci curvature.

1 Introduction

The Ricci curvature is one of the most important concepts in Riemannian geometry. In space physics, Ricci-flat manifolds represent vacuum solutions to an analogue of Einstein's equation for Riemannian manifolds with vanishing cosmological constant. They are used in the theory of general relativity. In mathematics, Calabi-Yau manifolds are Ricci-flat and can be applied to the superstring theory. There are some definitions of generalized Ricci curvature, and Ollivier's coarse Ricci curvature is one of them (see [4], [5]). It is formulated in terms of the Wasserstein distance on a metric space (X, d) with a random walk m . The coarse Ricci curvature is defined as, for $x, y \in X$,

$$\kappa(x, y) := 1 - \frac{W(m_x, m_y)}{d(x, y)},$$

where m_x and m_y are probability measures and $W(m_x, m_y)$ is the 1-Wasserstein distance between m_x and m_y .

On the other hand, the graph theory is used to model many types of relations and processes in physical, biological, social and information systems (see [8], [9], [10]). A graph $G = (V, E)$ is a pair of the set V of vertices and the set E of edges. If each edge is represented as an ordered pair of vertices, G is called a directed graph.

Lin-Lu-Yau modified Ollivier's definition of coarse Ricci curvature and defined the Ricci curvature on undirected graphs in [2]. In this paper, we define the Ricci curvature on *directed* graphs based on Lin-Lu-Yau's definition and calculate it on some directed graphs. As a main result, we classify directed Ricci-flat regular graphs (see Theorem 2.18).

2 Properties of Ricci curvature on directed graphs

2.1 Ricci curvature on directed graphs

Throughout the paper, let $G = (V, E)$ be a directed graph. For $x, y \in V$, we write (x, y) as an edge from x to y if any. We denote the set of vertices of G by $V(G)$ and the set of edges by $E(G)$.

Definition 2.1. (1) A *path* from vertex x to vertex y is a sequence of edges $(x, a_1), (a_1, a_2), \dots, (a_{n-2}, a_{n-1}), (a_{n-1}, y)$, where $n \geq 1$. We call n the *length* of the path.

(2) The *distance* $d(x, y)$ between two vertices $x, y \in V$ is the length of a shortest path connecting them.

Remark 2.2. The distance function has positivity and triangle inequality, but symmetry is not necessarily satisfied.

Definition 2.3. (1) For any $x \in V$, the *neighborhood* of x is defined as

$$\Gamma(x) := \Gamma^{\text{in}}(x) \cup \Gamma^{\text{out}}(x),$$

where $\Gamma^{\text{in}}(x) := \{y \mid (y, x) \in E\}$ and $\Gamma^{\text{out}}(x) := \{y \mid (x, y) \in E\}$.

(2) For all $x \in V$,

1. The *degree* of x , denoted by d_x , is the number of edges connecting x . i.e., $d_x = |\Gamma(x)|$.
2. The *in-degree* of x , denoted by d_x^{in} , is the number of edges with x as their terminal vertex. i.e., $d_x^{\text{in}} = |\Gamma^{\text{in}}(x)|$.
3. The *out-degree* of x , denoted by d_x^{out} , is the number of edges with x as their initial vertex. i.e., $d_x^{\text{out}} = |\Gamma^{\text{out}}(x)|$.

(3) G is a *regular graph* if every vertex has the same degree.

Definition 2.4. For a finite graph G , the *adjacency matrix* is defined by the following elements.

$$a_{i,j} = \begin{cases} 1, & (v_i, v_j) \in E, \\ 0, & (v_i, v_j) \notin E, \end{cases}$$

where $V(G) = \{v_1, v_2, \dots, v_n\}$.

Definition 2.5. If for every edge (x, y) , there exists a directed path from any $u \in \Gamma^{\text{out}}(x)$ to any $v \in \Gamma^{\text{out}}(y) \cup \{y\}$, then G is *locally strongly connected*.

In this paper, we assume that a directed graph G has the following properties.

1. Locally finiteness (every vertex has finite degree)

2. Simpleness (no loops and no multi-edges)
3. Locally strongly connectedness

Definition 2.6. For any vertex $x \in V(G)$ and any $\alpha \in [0, 1]$, a *probability measure* m_x is defined as

$$m_x^\alpha(v) = \begin{cases} \alpha + \frac{1-\alpha}{d_x} d_x^{\text{in}}, & v = x, \\ \frac{1-\alpha}{d_x}, & (x, v) \in E, \\ 0, & \text{otherwise.} \end{cases}$$

For m_x^α and m_y^α , the 1-Wasserstein distance between m_x^α and m_y^α is written by

$$W(m_x^\alpha, m_y^\alpha) = \inf_A \sum_{u,v \in V} A(u, v) d(u, v),$$

where $A : V \times V \rightarrow [0, 1]$ runs over all maps satisfying $\begin{cases} \sum_{v \in V} A(u, v) = m_x^\alpha(u), \\ \sum_{u \in V} A(u, v) = m_y^\alpha(v). \end{cases}$
The map A is called a *coupling* between m_x^α and m_y^α .

Remark 2.7. There exists a coupling A that attains the Wasserstein distance (see [6], [7]).

Definition 2.8. (1) For any $x, y \in V$, the α -Ricci curvature of x and y is defined as

$$\kappa_\alpha(x, y) = 1 - \frac{W(m_x^\alpha, m_y^\alpha)}{d(x, y)}.$$

(2) For any $x, y \in V$, the Ricci curvature of x and y is defined as

$$\kappa(x, y) = \lim_{\alpha \rightarrow 1} \frac{\kappa_\alpha(x, y)}{1 - \alpha}.$$

If $\kappa(x, y) = r$ holds for some constant $r \in \mathbb{R}$ and all edges $(x, y) \in E$, we say that G is a *graph of constant Ricci curvature*, and denote $\kappa(G) = r$. If $r = 0$, we say that G is *Ricci-flat*.

Proposition 2.9. For any two vertices x and y , $\kappa_\alpha(x, y)$ is concave in $\alpha \in [0, 1]$.

The proof is similar to the case of undirected graphs [2] and is omitted.

Proposition 2.10. For any two vertices x and y , we have

$$W(m_x^\alpha, m_y^\alpha) \geq \sup_{f \in F(G)} \sum_{z \in V} f(z) (m_x^\alpha(z) - m_y^\alpha(z)),$$

where $F(G) := \{f : V(G) \rightarrow \mathbb{R} \mid f(u) - f(v) \leq d(u, v)\}$.

Proof. For any coupling A , we have

$$\begin{aligned}
\sum_{u,v \in V} A(u,v)d(u,v) &\geq \sum_{u,v \in V} A(u,v)(f(u) - f(v)) \\
&= \sum_{u \in V} f(u) \sum_{v \in V} A(u,v) - \sum_{v \in V} f(v) \sum_{u \in V} A(u,v) \\
&= \sum_{u \in V} f(u)m_x^\alpha(u) - \sum_{v \in V} f(v)m_y^\alpha(v). \tag{2.1}
\end{aligned}$$

Since the left-hand side does not depend on f and the right-hand side does not depend on A , the proof is completed. \square

Proposition 2.11. For any two distinct vertices x, y , we have

$$\kappa_\alpha(x, y) \leq \frac{1-\alpha}{d(x,y)} \left\{ \frac{1}{d_y} \left(\sum_{k=1}^{d(x,y)-1} k|\Gamma_x^k(y)| - |\Gamma_x^+(y)| \right) + \frac{d_x^{\text{out}}}{d_x} \right\}, \tag{2.2}$$

where for all $1 \leq k \leq d(x, y)$,

$$\begin{aligned}
\Gamma_x^k(y) &= \{v \in \Gamma^{\text{out}}(y) \mid d(x, v) = d(x, y) - k\}, \\
\Gamma_x^+(y) &= \{v \in \Gamma^{\text{out}}(y) \mid d(x, v) = d(x, y) + 1\}.
\end{aligned}$$

Proof. For a fixed $x \in V$, we define $f(z) := -d(x, z)$. Then it follows that

$$\begin{aligned}
f(z) - f(w) &= -d(x, z) + d(x, w) \\
&\leq d(z, w).
\end{aligned}$$

Thus, $f \in F(G)$ holds. By Proposition 2.10, we have

$$\begin{aligned}
W(m_x^\alpha, m_y^\alpha) &\geq \sum_{z \in V} d(x, z)(m_y^\alpha(z) - m_x^\alpha(z)) \\
&= d(x, y)\left(\alpha + \frac{1-\alpha}{d_y}d_y^{\text{in}}\right) + \sum_{k=0}^{d(x,y)-1} (d(x, y) - k)|\Gamma_x^k(y)|\frac{1-\alpha}{d_y} \\
&\quad + (d(x, y) + 1)|\Gamma_x^+(y)|\frac{1-\alpha}{d_y} - \frac{d_x^{\text{out}}}{d_x}(1-\alpha) \\
&= d(x, y)\left(\alpha + \frac{1-\alpha}{d_y}d_y^{\text{in}}\right) - \frac{1-\alpha}{d_x}d_x^{\text{out}} \\
&\quad + d(x, y)\frac{1-\alpha}{d_y}d_y^{\text{out}} - \frac{1-\alpha}{d_y} \left(\sum_{k=1}^{d(x,y)-1} k|\Gamma_x^k(y)| - |\Gamma_x^+(y)| \right) \\
&= d(x, y) - \frac{1-\alpha}{d_y} \left(\sum_{k=1}^{d(x,y)-1} k|\Gamma_x^k(y)| - |\Gamma_x^+(y)| \right) - \frac{1-\alpha}{d_x}d_x^{\text{out}}.
\end{aligned}$$

This proves the proposition. \square

Remark 2.12. Proposition 2.9 implies that $h(\alpha) = \kappa_\alpha(x, y)/(1 - \alpha)$ is a monotone increasing function in $\alpha \in [0, 1)$ (the detail is written in the proof of Lemma 2.1 in [2]). Proposition 2.11 implies that $h(\alpha)$ is bounded. Thus, the limit $\kappa(x, y) = \lim_{\alpha \rightarrow 1} \kappa_\alpha(x, y)/(1 - \alpha)$ exists.

2.2 Examples

In this section, we treat some graphs on that a specific orientation is given, and calculate the Ricci curvature.

Example 2.13. Complete graph K_n

We consider a complete graph that is oriented by the following adjacency matrix A_n . If n is odd ($n = 2m + 1$), then elements of the adjacency matrix A_n are given as follows:

$$\begin{cases} a_{1,j} = 1, & j \in \{2, \dots, m+1\}, \\ a_{1,j} = 0, & j \in \{m+2, \dots, 2m+1\}, \\ a_{i,j} = 1, & i \in \{2, \dots, m+1\}, j \in \{1+i, \dots, m+i\}, \\ a_{i,j} = 0, & i \in \{2, \dots, m+1\}, j \notin \{1+i, \dots, m+i\}, \\ a_{i,j} = 1, & i \in \{m+2, \dots, 2m+1\}, j \in \{1+i, \dots, 2m+1\} \cup \{1, \dots, i-m\}, \\ a_{i,j} = 0, & i \in \{m+2, \dots, 2m+1\}, j \notin \{1+i, \dots, 2m+1\} \cup \{1, \dots, i-m\}. \end{cases}$$

For instance, A_3, A_5, A_7 are given by

$$A_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, A_5 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}, A_7 = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

If n is even ($n = 2m$), then we take away the $(2m+1)$ -th row and column from A_{2m+1} . The Ricci curvature depends on the edge, and the value is either one of the following.

Case $K_n = K_{2m+1}$

$$\kappa(x, y) \in \left\{ 0, \frac{1}{2m}, \dots, \frac{m-1}{2m} \right\}.$$

Case $K_n = K_{2m}$

$$\kappa(x, y) \in \left\{ \frac{1}{2m-1}, \dots, \frac{m-1}{2m-1} \right\}.$$

Example 2.14. Cycle C_n

We consider a cycle that is oriented as follows.
Let $V(C_n) = \{x_1, x_2, \dots, x_n\}$. For any $i \in \{1, 2, \dots, n-1\}$, $(x_i, x_{i+1}) \in E(C_n)$ and $(x_g, x_1) \in E(C_n)$. This is Ricci-flat, namely,

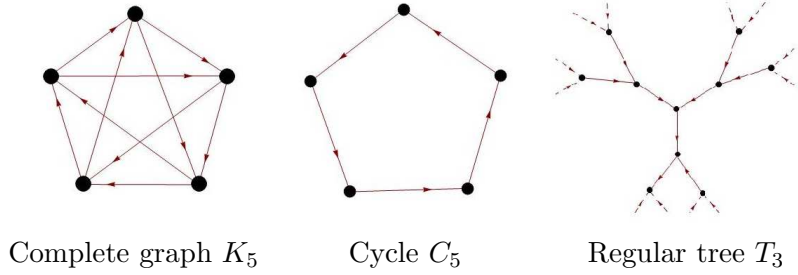
$$\kappa(x, y) = 0, \text{ for any } (x, y) \in E(C_n).$$

Example 2.15. Tree T

We consider a tree which is oriented so that for any $v \in V(T)$, $d_v^{\text{out}} = 1$.
Then the Ricci curvature is given by

$$\kappa(x, y) = \frac{1}{d_x} - \frac{1}{d_y}, \text{ for any } (x, y) \in E(T).$$

Remark 2.16. A tree T is Ricci-flat if and only if T is a regular tree.



2.3 Main theorem

Definition 2.17. G is called a *quasi-regular tree* qT_g if G consists of g directed regular trees starting from each vertex of a cycle C_g .

Theorem 2.18. Let G be a directed d -regular graph with at least one directed cycle.

If G satisfies the following conditions,

- (a) for any $(x, y) \in E(G)$, $\Gamma^{\text{out}}(x) \cap \Gamma^{\text{out}}(y) = \emptyset$,
- (b) for any two adjacent edges $(x, y), (y, z)$ of a cycle, $\Gamma^{\text{out}}(x) \cap \Gamma^{\text{in}}(z) = \{y\}$.

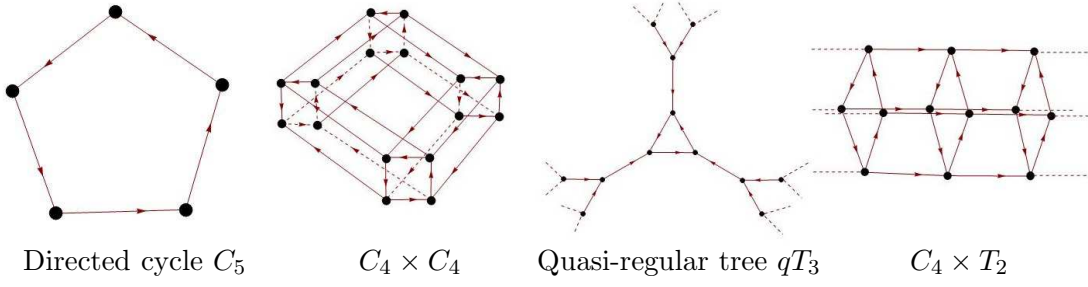
Then there exists an edge (x, y) of G such that

$$\kappa(x, y) \leq 0.$$

Moreover, G is Ricci-flat if and only if G is one of the following graphs:

- 1 Directed cycle C_g ,
- 2 Quasi-regular tree qT_g ,
- 3 $B \times A_1 \times A_2 \times \dots \times A_{n-1}$,

where n and g are constant, and n depends on d , B is either C_g or qT_g , and A_i is one of C_{h_i} , qT_{h_i} , or a regular tree ($i \in \{1, 2, \dots, n-1\}$, $g \geq h_i \geq 3$).



Proof. Since G has at least one directed cycle, we define g as follows:

$$g := \min \{ \text{the number of elements of } E(C) \mid C \text{ is a directed cycle of } G \} \geq 3.$$

We take a directed cycle C_g such that

$$V(C_g) := \{x_1, \dots, x_g\}.$$

Then we fix an edge $(x_1, x_2) \in E(C_g)$ and calculate the Ricci curvature $\kappa(x_1, x_2)$ according to the value of $d_{x_1}^{\text{out}}$.

Case 1 If $d_{x_1}^{\text{out}} = 1$, the probability measures of x_1 and x_2 are given by

$$m_{x_1}^\alpha(v) = \begin{cases} \alpha + \frac{1-\alpha}{d_{x_1}}(d_{x_1} - 1), & v = x_1, \\ \frac{1-\alpha}{d_{x_1}}, & v = x_2, \\ 0, & \text{otherwise,} \end{cases} \quad m_{x_2}^\alpha(v) = \begin{cases} \alpha + \frac{1-\alpha}{d_{x_2}}d_{x_2}^{\text{in}}, & v = x_2, \\ \frac{1-\alpha}{d_{x_2}}, & (x_2, v) \in E, \\ 0, & \text{otherwise.} \end{cases}$$

We define a coupling A between $m_{x_1}^\alpha$ and $m_{x_2}^\alpha$ by

$$A(u, v) = \begin{cases} \alpha + \frac{1-\alpha}{d_{x_2}}d_{x_2}^{\text{in}} - \frac{1-\alpha}{d_{x_1}}, & u = x_1, v = x_2, \\ \frac{1-\alpha}{d_{x_2}}, & u = x_1, v \in \Gamma^{\text{out}}(x_2), \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the Wasserstein distance between them satisfies

$$\begin{aligned} W(m_{x_1}^\alpha, m_{x_2}^\alpha) &\leq \alpha + \frac{1-\alpha}{d_{x_2}}d_{x_2}^{\text{in}} - \frac{1-\alpha}{d_{x_1}} + 2\frac{1-\alpha}{d_{x_2}}d_{x_2}^{\text{out}} \\ &= \alpha + (1-\alpha) + \frac{1-\alpha}{d_{x_2}}d_{x_2}^{\text{out}} - \frac{1-\alpha}{d_{x_1}}. \end{aligned}$$

Then for the Ricci curvature, we have

$$\kappa(x_1, x_2) \geq \frac{1}{d_{x_1}} - \frac{d_{x_2}^{\text{out}}}{d_{x_2}}. \tag{2.3}$$

On the other hand, we define a function $f : V(G) \rightarrow \mathbb{R}$ by

$$f(v) = \begin{cases} 1, & v = x_1, \\ -1, & v \in \Gamma^{\text{out}}(x_2), \\ 0, & \text{otherwise.} \end{cases}$$

Since it is clear that $f \in F(G)$, we obtain by Proposition 2.10

$$W(m_{x_1}^\alpha, m_{x_2}^\alpha) \geq \alpha + \frac{1-\alpha}{d_{x_1}}(d_{x_1} - 1) + \frac{1-\alpha}{d_{x_2}}d_{x_2}^{\text{out}}.$$

Then for the Ricci curvature, we have

$$\kappa(x_1, x_2) \leq \frac{1}{d_{x_1}} - \frac{d_{x_2}^{\text{out}}}{d_{x_2}},$$

and we obtain from (2.3),

$$\kappa(x_1, x_2) = \frac{1}{d_{x_1}} - \frac{d_{x_2}^{\text{out}}}{d_{x_2}}.$$

Suppose $\kappa(x_1, x_2) > 0$. Then it follows from the regularity of G that $d_{x_2}^{\text{out}} < 1$. However, since (x_2, x_3) is an edge of $E(C_g)$, we have $d_{x_2} \geq 1$, a contradiction. Thus we obtain $\kappa(x_1, x_2) \leq 0$, and $\kappa(x_1, x_2) = 0$ holds only when

$$d_{x_2}^{\text{out}} = 1. \tag{2.4}$$

Case 2 If $d_{x_1}^{\text{out}} \geq 2$, then we write $d_{x_1}^{\text{out}} = n_1, d_{x_2}^{\text{out}} = n_2, \Gamma^{\text{out}}(x_1) = \{x_2, u_1, u_2, \dots, u_{n_1-1}\}$, and $\Gamma^{\text{out}}(x_2) = \{x_3, v_1, v_2, \dots, v_{n_2-1}\}$.

Case 2-(1) If $n_1 < n_2 + 1$, then by the assumption and the condition (b), there exists an injective map $\psi : \Gamma^{\text{out}}(x_1) \rightarrow \Gamma^{\text{out}}(x_2)$ such that for all u_i , it holds

$$L'_i := d(u_i, \psi(u_i)) < \infty.$$

Since $d_{x_1}^{\text{in}} = d - n_1, d_{x_2}^{\text{in}} = d - n_2$, the probability measures of x_1 and x_2 are given by

$$m_{x_1}^\alpha(v) = \begin{cases} \alpha + \frac{1-\alpha}{d}(d - n_1), & v = x_1, \\ \frac{1-\alpha}{d}, & v \in \Gamma^{\text{out}}(x_1), \\ 0, & \text{otherwise} \end{cases}, \quad m_{x_2}^\alpha(v) = \begin{cases} \alpha + \frac{1-\alpha}{d}(d - n_2), & v = x_2, \\ \frac{1-\alpha}{d}, & v \in \Gamma^{\text{out}}(x_2), \\ 0, & \text{otherwise.} \end{cases}$$

We define a coupling A $m_{x_1}^\alpha$ and $m_{x_2}^\alpha$ by

$$A(u, v) = \begin{cases} \alpha + \frac{1-\alpha}{d}(d - n_1), & u = x_1, v = x_2, \\ \frac{1-\alpha}{d}, & u \in \Gamma^{\text{out}}(x_1), v \in \Gamma^{\text{out}}(x_2) \cup \{y\}, \\ 0, & \text{otherwise.} \end{cases}$$

Thus the Wasserstein distance between them satisfies

$$\begin{aligned} W(m_{x_1}^\alpha, m_{x_2}^\alpha) &\leq \alpha + \frac{1-\alpha}{d}(d - n_2 - 1) + \sum_{i=1}^{n_1-1} L'_i \frac{1-\alpha}{d} + 2\frac{1-\alpha}{d}(n_2 + 1 - n_1) \\ &= 1 - \frac{1-\alpha}{d} \left(n_2 - 2n_1 + 1 + \sum_{i=1}^{n_1-1} L'_i \right), \end{aligned}$$

and we obtain

$$\kappa(x_1, x_2) \geq \frac{1}{d} \left(2n_1 - n_2 - 1 - \sum_{i=1}^{n_1-1} L'_i \right). \quad (2.5)$$

On the other hand, we define a function $f : V(G) \rightarrow \mathbb{R}$ by

$$f(v) = \begin{cases} 1, & v = x_1, \\ -1, & v \in \Gamma^{\text{out}}(x_2), \\ L'_i - 1, & v \in \Gamma^{\text{out}}(x_1) \setminus \{x_2\}, \\ 0, & \text{otherwise.} \end{cases}$$

By the Proposition 2.10, we obtain

$$\begin{aligned} W(m_{x_1}^\alpha, m_{x_2}^\alpha) &\geq \alpha + \frac{1-\alpha}{d}(d - n_1) + n_2 \frac{1-\alpha}{d} + \sum_{i=1}^{n_1-1} (L'_i - 1) \frac{1-\alpha}{d} \\ &= 1 - 2n_1 \frac{1-\alpha}{d} + n_2 \frac{1-\alpha}{d} + \sum_{i=1}^{n_1-1} L'_i \frac{1-\alpha}{d} - \frac{1-\alpha}{d}. \end{aligned}$$

Thus by (2.5), the Ricci curvature satisfies

$$\begin{aligned} \kappa(x_1, x_2) &= \frac{1}{d} \left(2n_1 - n_2 - 1 - \sum_{i=1}^{n_1-1} L'_i \right), \\ &\leq \frac{1}{d} \{ 2n_1 - n_2 - 1 - (n_1 - 1) \} = \frac{1}{d} (n_1 - n_2). \end{aligned}$$

Since $L'_i \geq 1$ holds for all i by the assumption, the second inequality is obtained. Therefore, $\kappa(x_1, x_2) \leq 0$ holds, and $\kappa(x_1, x_2) = 0$ implies $n_1 = n_2$.

Case 2-(2) If $n_1 \geq n_2 + 1$, then by the assumption and by the condition (b), there exists an injective map $\phi : \Gamma^{\text{out}}(x_2) \rightarrow \Gamma^{\text{out}}(x_1)$ such that for all v_i , it holds that

$$L_i := d(\phi(v_i), v_i) < \infty,$$

and for each $u'_i \in \Gamma^{\text{out}}(x_1) \setminus \phi(\Gamma^{\text{out}}(x_2))$, it holds that

$$l_i := d(u'_i, y) < \infty.$$

The probability measures are the same as in case 2-(1), and a coupling A between them is taken to be similar to case 2-(1). Thus the Wasserstein distance between them satisfies

$$\begin{aligned} W(m_{x_1}^\alpha, m_{x_2}^\alpha) &\leq \alpha + \frac{1-\alpha}{d}(d-n_1) + \sum_{i=1}^{n_1-n_2-1} l_i \frac{1-\alpha}{d} + \sum_{i=1}^{n_2} L_i \frac{1-\alpha}{d} \\ &= 1 - n_1 \frac{1-\alpha}{d} + \sum_{i=1}^{n_1-n_2-1} l_i \frac{1-\alpha}{d} + \sum_{i=1}^{n_2} L_i \frac{1-\alpha}{d}. \end{aligned}$$

Therefore, for the Ricci curvature, we have

$$\kappa(x_1, x_2) \geq \frac{1}{d} \left(n_1 - \sum_{i=1}^{n_1-n_2-1} l_i - \sum_{i=1}^{n_2} L_i \right). \quad (2.6)$$

On the other hand, we define a function $f : V(G) \rightarrow \mathbb{R}$ by

$$f(v) = \begin{cases} 1, & v = x_1, \\ -1, & v \in \Gamma^{\text{out}}(x_2), \\ L_i - 1, & v = u_i \in \phi(\Gamma^{\text{out}}(x_2)), \\ l_i, & v = u'_i \in \Gamma^{\text{out}}(x_1) \setminus \phi(\Gamma^{\text{out}}(x_2)), \\ 0, & \text{otherwise.} \end{cases}$$

It follows from the definition of L_i and l_i that $d(u_i, x_1) \geq L_i - 2$ and $d(u'_i, x) \geq l_i - 1$, and so $f \in F(G)$. By Proposition 2.10, we obtain

$$\begin{aligned} W(m_{x_1}^\alpha, m_{x_2}^\alpha) &\geq \alpha + \frac{1-\alpha}{d}(d-n_1) + n_2 \frac{1-\alpha}{d} + \sum_{i=1}^{n_2} (L_i - 1) \frac{1-\alpha}{d} + \sum_{i=1}^{n_1-n_2-1} l_i \frac{1-\alpha}{d} \\ &= 1 - n_1 \frac{1-\alpha}{d} + \sum_{i=1}^{n_2} L_i \frac{1-\alpha}{d} + \sum_{i=1}^{n_1-n_2-1} l_i \frac{1-\alpha}{d}. \end{aligned}$$

Therefore, by (2.6), the Ricci curvature satisfies

$$\kappa(x_1, x_2) = \frac{1}{d} \left(n_1 - \sum_{i=1}^{n_1-n_2-1} l_i - \sum_{i=1}^{n_2} L_i \right).$$

By the condition (a), $L_i \geq 1$ and $l_i \geq 1$ hold for all i . Thus, the necessary and sufficient condition for $\kappa(x_1, x_2) > 0$ is

$$L_i = 1, \quad l_i = 1, \quad \text{for all } i. \quad (2.7)$$

The necessary and sufficient condition for $\kappa(x_1, x_2) = 0$ is either one of the following.

(1) There exists $i_0 \in \{1, \dots, n_2\}$ such that

$$\begin{cases} L_{i_0} = 2, \\ L_i = 1, & \text{for any } i \in \{1, \dots, n_2\} \setminus \{i_0\}, \\ l_j = 1, & \text{for any } j \in \{1, \dots, n_1 - n_2 - 1\}. \end{cases}$$

(2) There exists $j_0 \in \{1, \dots, n_1 - n_2 - 1\}$ such that

$$\begin{cases} l_{j_0} = 2, \\ l_j = 1, & \text{for any } j \in \{1, \dots, n_1 - n_2 - 1\} \setminus \{j_0\}, \\ L_i = 1, & \text{for any } i \in \{1, \dots, n_2\}. \end{cases}$$

Thus, $\kappa(x_1, x_2) > 0$ holds only when $n_1 \geq n_2 + 1$. Applying the same argument to the other edges on C_g , we obtain

$$n_1 \geq n_2 + 1 \geq n_3 + 2 \geq \dots \geq n_g + g - 1 \geq n_1 + g,$$

a contradiction. This implies that there exists an edge (x, y) such that

$$\kappa(x, y) \leq 0.$$

On the other hand, in addition to 2-(1), the necessary and sufficient condition for $\kappa(x_1, x_2) = 0$ is $n_2 \leq n_1$. In the same way as in the above argument for other edges on C_g , we obtain

$$n_1 \leq n_g \leq n_{g-1} \leq \dots \leq n_2 \leq n_1,$$

which implies

$$n := n_1 = n_2 = \dots = n_g.$$

Thus $\kappa(x_1, x_2) = 0$ implies that there exists a bijective map $\phi : \Gamma^{\text{out}}(x_1) \rightarrow \Gamma^{\text{out}}(x_2)$ such that

$$\begin{cases} \phi(x_2) = x_3, \\ L_i := d(u_i, \phi(u_i)) = 1, & \text{for all } i \in \{1, 2, \dots, n-1\}. \end{cases}$$

Based on the above argument, we construct a regular Ricci-flat directed graph G .

Case $n = 1$

If $d_{x_1}^{\text{in}} = 1$, then it is clear that G must be a directed cycle. If $d_{x_1}^{\text{in}} > 1$, then there exists $a_i \in V \setminus \{x_{i+1}\}$ such that $(a_i, x_i) \in E$. There are $d - 1$ edges coming in a_i provided we consider the case of $d_{a_i}^{\text{out}} = 1$ for all a_i . Then, we obtain a quasi-regular tree qT_g . On the other hand, we suppose that for all a_i , $d_{a_i}^{\text{out}} > 1$ holds. For the edge (a_1, x_1) , by the necessary and sufficient condition to be Ricci-flat and by the condition (a), there exists $b_1 \in V$ such that

$$(a_1, b_1), (b_1, a_2) \in E,$$

and b_i is defined in the same way as b_1 . Then, a subgraph H is defined as

$$\begin{aligned} V(H) &:= \{a_1, \dots, a_g, b_1, \dots, b_g\}, \\ E(H) &:= \{(a_1, b_1), (b_1, a_2), \dots, (a_{g-1}, b_g), (b_g, a_1)\}, \end{aligned}$$

i.e., H is a directed cycle. For the edge (b_1, a_2) , by the necessary and sufficient condition to be Ricci-flat, there exists $c_1 \in V \setminus \{b_1\}$ such that

$$(b_1, c_1), (c_1, b_2) \in E,$$

contradicting the condition (b).

Case $n = 2$

For all $i \in \{1, 2, \dots, g-1\}$, there exist vertices x'_i and x'_g satisfying

$$\Gamma^{\text{out}}(x_i) \setminus \{x_{i+1}\} = \{x'_i\}$$

and $\Gamma^{\text{out}}(x_g) \setminus \{x_1\} = \{x'_g\}$. For all $i \in \{1, 2, \dots, g-1\}$, by the necessary and sufficient condition to be Ricci-flat and by the condition (b), $(x'_i, x'_{i+1}) \in E$ and $(x'_g, x'_1) \in E$. Then a subgraph G' is defined as

$$\begin{aligned} V(G') &:= \{x'_1, x'_2, \dots, x'_g\}, \\ E(G') &:= \{(x'_1, x'_2), (x'_2, x'_3), \dots, (x'_g, x'_1)\}, \end{aligned}$$

i.e., G' is a directed cycle. Based on the G' , we construct G . First, we suppose $d_{x_1}^{\text{in}} = 1$, namely, G is a 3-regular graph. On the other hand, although $d_{x_1} = d_{x'_1} = 3$, $\kappa(x_1, x'_1) \neq 0$ follows, a contradiction. We consider the case $d_{x_1}^{\text{in}} = 2$, namely, G is a 4-regular graph. For $i \in \{2, \dots, g\}$, there exist vertices x_i^1 and x_1^1 satisfying $x_i^1 \in \Gamma^{\text{in}}(x_i) \setminus \{x_{i-1}\}$ and $x_1^1 \in \Gamma^{\text{in}}(x_1) \setminus \{x_g\}$. For $\kappa(x_1^1, x_1)$ to be Ricci-flat and to satisfy condition (a), we get 2 patterns of connection of edges as follows.

There exists $y_1 \in V \setminus \{x_1\}$ such that (x_1^1, y_1) and $(y_1, x'_1) \in E$.

There exists $(x_1^1, x_2^1) \in E$.

In the first case,

$$d_{x_1} = d_{x'_1} = 4,$$

then $\kappa(x_1, x'_1) \neq 0$ is obtained, but this is a contradiction. Thus the second case occurs.

Now a subgraph G^1 defined as

$$\begin{aligned} V(G^1) &:= \{x_1^1, x_2^1, \dots, x_g^1\}, \\ E(G^1) &:= \{(x_1^1, x_2^1), (x_2^1, x_3^1), \dots, (x_g^1, x_1^1)\} \end{aligned}$$

is a directed cycle. Next, for a graph G^1 , applying the same argument as G , we have a directed cycle G^2 ($V(G^2) = (x_1^2, x_2^2, \dots, x_g^2)$). Repeating this, we obtain the following graph G . If there exists $h \in \mathbb{Z}$ such that $x_i^h = x'_i$, $G = C_g \times C_h$. If it holds $x_i^h \neq x'_i$ for all $h \in \mathbb{Z}$, $G = C_g \times T_2$. The case of $d_{x_1}^{\text{in}} > 2$ is similarly discussed. If we orient the edges, we remark the following.

Remark 2.19. Suppose that there exist $\bar{x}_i \in \Gamma^{\text{out}}(x_i) \setminus \{x_{i+1}\}$, $i \in \{1, \dots, g-1\}$, and $\bar{x}_g \in \Gamma^{\text{out}}(x_g) \setminus \{x_1\}$ satisfying

$$\begin{cases} (\bar{x}_i, \bar{x}_{i+1}) \in E, (\bar{x}_g, \bar{x}_1) \in E, & i \in \{1, \dots, g-1\}, \\ \bar{x}_j = \bar{x}_{j+g/2}, & j \in \{1, \dots, g/2\}. \end{cases}$$

Then we construct a directed cycle C' as follows.

$$\begin{aligned} V(C') &:= \{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{g/2}\}, \\ E(C') &:= \{(\bar{x}_1, \bar{x}_2), (\bar{x}_2, \bar{x}_3), \dots, (\bar{x}_{g/2}, \bar{x}_1)\}. \end{aligned}$$

Thus the length of C' is $g/2$, but this contradicts the definition of g .

As a result, G is one of the following graphs.

1. $C_g \times \text{regular tree}$
2. $qT_g \times \text{regular tree}$
3. $qT_g \times C_h$
4. $qT_g \times qT_h$

Case $n > 2$

We consider the same argument as in the case $n = 2$. There exists no Ricci-flat graph satisfying $d_{x_1}^{\text{in}} < n$. If we consider the case $d_{x_1}^{\text{in}} = n$, then we obtain

$$G = C_g \times A_1 \times A_2 \times \dots \times A_{n-1},$$

where $A_i = C_{h_i}$ ($g \geq h_i \geq 3$) or T_2 for $i \in \{1, 2, \dots, n-1\}$. If we consider the case $d_{x_1}^{\text{in}} > n$, then we get

$$G = B \times A_1 \times A_2 \times \dots \times A_{n-1},$$

where $B = C_g$ or qT_g , and for $i \in \{1, 2, \dots, n-1\}$, $A_i = C_{h_i}$, qT_{h_i} ($g \geq h_i \geq 3$) or a regular tree. \square

Remark 2.20. In the present paper, we assume several conditions to simplify the calculation of the Ricci curvature. Without these conditions, cases to consider become too big, and it is hard to classify Ricci-flat graphs. In the case of undirected graphs, Lin-Lu-Yau classified the Ricci-flat graphs in [1].

References

- [1] Y. Lin, and L. Lu, *Ricci-flat graphs with girth at least five*, arXiv:1301.0102 (2013).
- [2] Y. Lin, L. Lu and S. T. Yau, *Ricci curvature of graphs*, Tohoku Math. J. *63* (2011) 605-627.
- [3] Y. Lin and S. T. Yau, *Ricci Curvature and eigenvalue estimate on locally finite graphs*, Math. Res. Lett. *17* (2010) 343-356.
- [4] Y. Ollivier, *Ricci curvature of Markov chains on metric spaces*, J. Functional Analysis. *256* (2009) 810-864.
- [5] Y. Ollivier, *A survey of Ricci curvature for metric space and Markov chains*, Probabilistic approach to geometry *57* (2010) 343-381.
- [6] C. Villani, *Topics in Mass Transportation*, Graduate Studies in Mathematics, Amer. Mathematical Society *58* (2003).
- [7] C. Villani, *Optimal transport, Old and new*, Grundlehren der Mathematischen Wissenschaften *338*, Springer, Berlin (2009).
- [8] T. Washio, and H. Motoda. *State of the art of graph-based data mining*, Acm Sigkdd Explorations Newsletter *5.1* (2003) 59-68.
- [9] D. J. Watts and S. H. Strogatz, *Collective dynamics of ‘small-world’ networks* Nature *393* (1998) 440-442.
- [10] W. Xindong, et al., *Top 10 algorithms in data mining*, Knowledge and Information Systems *14.1* (2008) 1-37.

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